

Raychaudhuri equation in the Finsler-Randers spacetime and Generalized scalar-tensor theories

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December 15, 2016

Abstract

In this work, we obtain the Raychaudhuri equations for various types of Finsler spaces as the Finsler-Randers (FR) space-time and in a more general geometrical structure of the space-time manifold which contains two fibres that are two scalars which represent inflaton fields $\phi^{(1)}, \phi^{(2)}$. In addition, the energy-conditions are studied in a FR cosmology and are correlated with FRW model. Finally an application of Raychaudhuri equation for the model $M \times \{\phi^{(1)}\} \times \{\phi^{(2)}\}$ with M a FRW space is given.

Keywords: Finsler Geometry, Raychaudhuri Equation, Scalar-Tensor Theories,

1 Introduction

Raychaudhuri equation has been introduced by A. Raychaudhuri [1]. It is a fundamental equation in gravitation and cosmology and has been studied and generalized in many ways, in different cases [2] - [19]. In Finsler space-time this equation was introduced in [20], [21], [22] and later studied in a different way [23]. Raychaudhuri equation describes the evolution of a gravitating fluid and provides a validation of our expectation that gravitation should be a universal attractive force between any two particles in general relativity. When cosmological fields such as dark energy, electromagnetic fields, spurionic fields or anisotropy are considered in the conventional Raychaudhuri equation, the evolution of the acceleration of the Universe is affected [24]. The form of the Raychaudhuri equation varies when the metrical structure of the space changes for example in spaces with torsion or generalized metric spaces such as a Finsler space time in which the Friedmann equations are modified [20], [21], [25]. In the case of null geodesics, the null Raychaudhuri equation, plays a key role in geometrical optics of a curved space-time. In addition, the Raychaudhuri equation ensures the existence of conjugate points with the famous singularity theorems in congruences and provides that the strong energy condition holds [26]. The Raychaudhuri equation is produced by the structure of deviation equation of nearby geodesics or curves and it is considered as an index that records the evolution of the geodesics (focusing/defocusing) and plays a fundamental role in the dynamics of the fluid. The effects of gravity are encoded in the evolution of the expansion which is governed by Raychaudhuri's equations. The different types of Raychaudhuri's equation are useful when we want to pass to different geometrical structures and phases during the evolution of the universe. All these types can describe the corresponding changes of the equations of motion. The Raychaudhuri equation comes from the deviation of geodesics and is related with the tidal force fields ($R_{jkl}^i \neq 0$). Consequently, the

gravitational field is transferred in it and the tidal field Ricci is included in the equation. When we consider a congruence of geodesics representing the motion of flow lines they pass through a surface which is vertical to the fluid. In this case the accelerated expansion of the universe can be studied through its effects on a congruence of geodesics [27]. While the elementary particles are moving to adjacent geodesics of the fluid a deviation happens due to the gravitational field and torsion in generalized metric spaces. The motion and the variation of the geodesic lines after some time causes a distortion of the fluid surface. Thus the variation of the volume (expansion/contraction) includes tidal field, rotation and shear which are connected to the deviation vector. Changes in the deviation vector between two nearby world flow lines are monitored by Raychaudhuri's equation, which is closely related to the evolution of the universe.

The deviation of geodesics is of fundamental significance in the general relativity and gravitation because it interrelates the interaction of curvature with the matter and plays the role of a code for the tidal forces. The deviation of geodesics in generalized metric spaces of Finsler structure has been studied by H. Rund [28] and E. Cartan [29] and later in a series of papers for Finsler and Lagrange spaces and their applications in Finslerian space-time [30], [31], [32], [33].

A fundamental factor for the form of Raychaudhuri equation is obtained by considering the structure of the space and the kind of the curvature. Especially, in generalized metric spaces such as a Finsler spacetime where the curvatures are more than one. The concept of the Raychaudhuri equation is extended with the dependence of internal variables as the velocity (scalars, spinors or an anisotropic field) and extra terms. In that case, the Raychaudhuri equations are expressed with curvatures of the form $K_{jkl}^i(x, v)$, $S_{\beta\gamma\delta}^\alpha(x, v)$ coupling with a direction y or a velocity vector v . In a different way the authors in [10] have studied a coupling between velocity and the Riemann curvature tensor. The dependence of curvatures on the direction (velocity) of a scalar fields is a consequence of the local anisotropy of the spacetime which is intrinsically considered in the space. So in an effective theory of modified gravity more general than the Einstein one, the geodesic deviation equation and the Raychaudhuri equation can include the effects of curvature-matter-velocity coupling. Furthermore, an anisotropic form of curvature has been considered in [24] for astrophysical considerations.

A type of Finsler space is the Finsler-Randers space hereafter (FR) [34], which constitutes an important geometrical structure in Finsler spaces [35], [40] as far as its applications. In the general relativity and cosmology are concerned [25], [51], [52]. In this case, the equation of geodesics of Finsler space in general is the equation of motion of a charged particle moving inside an electromagnetic and gravitational field and it is connected to the Lorentz force. In a FR space we can replace on the second part (one form) the electromagnetic potential with a quantity that represents a field of anisotropy, a scalar, or spinning particles. In the classical type of FR space-time the geometry contains information for the gravity and the electromagnetism. Of course that is not a complete and self-consistent theory of unification. However, one may not overlook the fact that it gives us a picture of what we should expect from a theory of unified field in a more extended geometrical framework.

The paper is organized as follows: In section 2 we give in brief some geometrical concepts from the theory of Finsler geometry. In section 3 we present the Raychaudhuri equation in the Finslerian spacetime and we give its form for a FR space. The energy conditions are examined in this model. In addition, bounce conditions are presented and they are compared with the FRW cosmology. Finally in section 4 two forms of the Raychaudhuri equations are studied in a generalized scalar-tensor theory with scalars $\phi^{(1)}, \phi^{(2)}$ that play the role of fibres. Some concluding remarks are given in section 5.

2 Preliminaries

In the following, we briefly present fundamental geometrical concepts from the theory of Finsler spaces [38]. We consider a smooth 4-dimensional manifold M , (TM, π, M) , its tangent bundle, and $T\tilde{M} = TM \setminus 0$, where 0 means the image of the null cross-section of the projection $\pi : TM \rightarrow M$. We also consider a local system of coordinates $x^i, i = 0, 1, 2, 3$ and U , a chart of M . Then the couple (x^i, y^a) is a local coordinate system on $\pi^{-1}(U)$ in TM . A coordinate transformation on the total space TM is given by

$$\tilde{x}^i = \tilde{x}^i(x^0, \dots, x^3) \quad (1)$$

$$\det \left\| \frac{\partial \tilde{x}^i}{\partial x^j} \right\| \neq 0 \quad (2)$$

$$\tilde{y}^a = \frac{\partial \tilde{x}^a}{\partial x^b} y^b \quad (3)$$

$$x^\alpha = \delta_i^\alpha x^i \quad (4)$$

A Finsler metric on M is a function $F : TM \rightarrow \mathbb{R}$ having the properties:

1. The restriction of F to $T\tilde{M}$ is of class C^∞ , and F is only continuous on the image of the null cross-section in the tangent bundle to M .
2. F is positively homogeneous of degree 1 with respect to (y^α) , $F(x, ky) = kF(x, y), k \in \mathbb{R}_+^*$.
3. The quadratic form on \mathbb{R}^n with the coefficients $f_{ij} = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}$ defined on $T\tilde{M}$ is non-degenerate ($\det(f_{ij}) \neq 0$), with rank $(f_{ij}) = 4$.

A non-linear connection N on TM is a distribution on TM , supplementary to the vertical distribution V on TM :

$$T_{(x,y)}(TM) = N_{(x,y)} \oplus V_{(x,y)}. \quad (5)$$

A non-linear connection can be defined as

$$N_j^a = \frac{\partial G^a}{\partial y^j} \quad (6)$$

where G^a are defined by

$$G^a = \frac{1}{4} f^{aj} \left(\frac{\partial^2 F}{\partial y^j \partial x^k} y^k - \partial_j F \right), \quad (7)$$

and the relation

$$\frac{dy^a}{ds} + 2G^a(x, y) = 0 \quad (8)$$

follows from the Euler-Lagrange equations

$$\frac{d}{ds} \left(\frac{\partial F}{\partial y^a} \right) - \frac{\partial F}{\partial x^a} = 0. \quad (9)$$

The transformation rule of the non-linear connection coefficients is given by

$$\tilde{N}_i^a = \frac{\partial \tilde{x}^a}{\partial x^b} \frac{\partial x^j}{\partial \tilde{x}^i} N_j^b(x, y) + \frac{\partial \tilde{x}^a}{\partial x^h} \frac{\partial^2 x^h}{\partial \tilde{x}^i \partial \tilde{x}^b} y^b \quad (10)$$

also,

$$\frac{\delta}{\delta \tilde{x}^i} = \frac{\partial x^j}{\partial \tilde{x}^i} \frac{\delta}{\delta x^j}, \quad \frac{\partial}{\partial \tilde{y}^a} = \frac{\partial x^b}{\partial \tilde{x}^a} \frac{\partial}{\partial y^b} \quad (11)$$

$$d\tilde{x}^i = \frac{\partial \tilde{x}^i}{\partial x^j} dx^j, \quad \delta \tilde{y}^\alpha = \frac{\partial \tilde{x}^\alpha}{\partial x^b} \delta y^b \quad (12)$$

A local basis of $T_{(x,y)}(TM)$, $(\delta_i, \dot{\partial}_\alpha)$ adapted to the horizontal distribution N has the form

$$\delta_i = \partial_i - N_i^\alpha(x, y) \dot{\partial}_\alpha, \quad (13)$$

where

$$\partial_i = \frac{\partial}{\partial x^i}, \quad \dot{\partial}_\alpha = \frac{\partial}{\partial y^\alpha} \quad (14)$$

and $N_i^\alpha(x, y)$ are the coefficients of the non-linear Cartan connection N. The concept of non-linear connection is fundamental in the geometry of vector bundles and locally anisotropic spaces. It is a powerful tool for a geometrical unification of fields. For example, in the case of the gravitational field, the non-linear connection in the framework of tangent bundle unifies the external and internal spaces, i.e., the position space (the base manifold M) with the tangent space $T_p M$. In other words, it is connected with the local anisotropic structure of space-time (depends on the velocities). The dual local basis is

$$\{d^i = dx^i, \delta^\alpha = \delta y^\alpha = dy^\alpha + N_j^\alpha dx^j\}_{i,\alpha=\overline{0,3}} \quad (15)$$

A d-connection on the tangent bundle TM of spacetime is a linear connection on TM which preserves, by parallelism, the horizontal distribution N and the vertical distribution V on TM. A covariant derivative associated with a d-connection becomes d-covariant. Generally, an h-v metric on the tangent bundle (TM, π , M) is given by

$$G = f_{ij}(x, y) dx^i \otimes dx^j + h_{\alpha\beta} \delta y^\alpha \otimes \delta y^\beta. \quad (16)$$

We consider a metrical d-connection $CT = (N_j^\alpha, L_{jk}^i, C_{jk}^i)$ with the property

$$f_{ij|k} = \delta_k f_{ij} - L_{ik}^h f_{hj} - L_{jk}^h f_{ih} = 0, \quad (17)$$

$$f_{ij|k} = \dot{\partial}_k f_{ij} - C_{ik}^h f_{hj} - C_{jk}^h f_{ih} = 0 \quad (18)$$

where

$$L_{jk}^i = \frac{1}{2} f^{ir} (\delta_j f_{rk} + \delta_k f_{jr} - \delta_r f_{jk}) \quad (19)$$

$$C_{jk}^i = \frac{1}{2} f^{ir} (\dot{\partial}_j f_{rk} + \dot{\partial}_k f_{jr} - \dot{\partial}_r f_{jk}). \quad (20)$$

The coordinate transformation of the objects L_{jk}^i and C_{jk}^i is

$$\tilde{L}_{jk}^i = \frac{\partial \tilde{x}^i}{\partial x^h} \frac{\partial x^l}{\partial \tilde{x}^j} \frac{\partial x^r}{\partial \tilde{x}^k} L_{lr}^h(x, y) + \frac{\partial \tilde{x}^i}{\partial x^r} \frac{\partial^2 x^r}{\partial \tilde{x}^j \partial \tilde{x}^k} \quad (21)$$

$$\tilde{C}_{jk}^i = \frac{\partial \tilde{x}^i}{\partial x^h} \frac{\partial x^l}{\partial \tilde{x}^j} \frac{\partial x^r}{\partial \tilde{x}^k} C_{lr}^h(x, y) \quad (22)$$

The Cartan torsion coefficients C_{ijk} are given by

$$C_{ijk} = \frac{1}{2} \dot{\partial}_k f_{ij} \quad (23)$$

while the Christoffel symbols of the first and second kind for the metric f_{ij} are

$$\gamma_{ijk} = \frac{1}{2} \left(\frac{\partial f_{kj}}{\partial x^i} + \frac{\partial f_{ik}}{\partial x^j} - \frac{\partial f_{ij}}{\partial x^k} \right) \quad (24)$$

$$\gamma_{ij}^l = \frac{1}{2} f^{lk} \left(\frac{\partial f_{kj}}{\partial x^i} + \frac{\partial f_{ik}}{\partial x^j} - \frac{\partial f_{ij}}{\partial x^k} \right), \quad (25)$$

respectively. The torsions and curvatures which we use are given by

$$T_{kj}^i = 0, \quad S_{kj}^i = 0 \quad (26)$$

$$R_{jk}^i = \delta_k N_j^i - \delta_j N_k^i, \quad P_{jk}^i = \dot{\partial}_k N_j^i - L_{kj}^i \quad (27)$$

$$P_{jk}^i = f^{im} P_{mjk}, \quad P_{ijk} = C_{ijk|ly}^l \quad (28)$$

$$R_{jkl}^i = \delta_l L_{jk}^i + \delta_k L_{jl}^i + L_{jk}^h L_{hl}^i - L_{jl}^h L_{hk}^i + C_{jc}^i R_{kl}^c \quad (29)$$

$$S_{jikh} = C_{iks} C_{jh}^s - C_{ih}s C_{jk}^s \quad (30)$$

$$P_{ihkj} = C_{ijk|h} - C_{hjk|i} + C_{hj}^r C_{rik|ly}^l - C_{ij}^r C_{rkh|ly}^l \quad (31)$$

$$S_{ikh}^l = f^{lj} S_{jikh} \quad (32)$$

$$P_{ikh}^l = f^{lj} P_{jikh} \quad (33)$$

The Ricci identities for the d-connection are

$$X^i|_k|_h - X^i|_h|_k = X^r R_r^i{}_{kh} - X^i|_r R^r{}_{kh}, \quad (34)$$

$$X^i|_k|_h - X^i|_h|_k = X^r P_r^i{}_{kh} - X^i|_r C^r{}_{kh} - X^i|_r P^r{}_{kh} \quad (35)$$

$$X^i|_k|_h - X^i|_h|_k = X^r S_r^i{}_{kh} \quad (36)$$

3 Raychaudhuri Equation

The Raychaudhuri equations are concerned with the kinematics of flows. Flows are generated by a vector field, they are the integral curves of a given vector field. These curves may be geodesics or non-geodesics. A flow is a congruence of curves that are time-like, null or sometimes space-like. We are more interested in deriving additional kinematic characteristics of such flows with extra terms. The evolution equations (along the flows) of quantities that characterize the flow in a given background spacetime are the Raychaudhuri equations. In fluid flows of cosmology there is a preferred 4-velocity vector field $u^a : u^a u_a = 1$ that represents the average motion of matter. Let τ be the proper time along these world lines $u^a = \frac{dx^a}{d\tau}$. The acceleration vector of u^a is $\dot{u}^a = u_{;b}^a u^b$, which vanishes if and only if the flow lines are geodesics for a pseudo-Riemannian background.

The covariant differentiation of the velocity field u is a second rank tensor

$$\nabla_b u_a = \sigma_{ab} + \omega_{ab} + \frac{1}{3} h_{ab} \Theta$$

which is split into three parts : the symmetric traceless part, the shear of the flow

$$\sigma_{ab} = \frac{1}{2} (\nabla_b u_a + \nabla_a u_b) - \left(\frac{1}{3}\right) h_{ab} \Theta,$$

the antisymmetric part, the rotation of the flow

$$\omega_{ab} = \frac{1}{2} (\nabla_b u_a - \nabla_a u_b)$$

and the trace, the expansion of the flow

$$\Theta = \nabla_a u^a.$$

The Raychaudhuri equation, giving the evolution of Θ along the fluid flow lines. Thus one has

$$\dot{\Theta} + \frac{1}{3}\Theta^2 = 2(\omega^2 - \sigma^2) + \dot{u}_{;a}^a - \frac{1}{2}\kappa(\mu + 3p) + \Lambda \quad (37)$$

This is the generic form of the Raychaudhuri equation, which is the fundamental equation of gravitational attraction and shows that shear, energy density and pressure tend to make matter collapse or decelerate the expansion while vorticity and a positive cosmological constant tend to make matter expand.

3.1 Raychaudhuri Equation in a Finsler Space-time

Finslerian congruences

Suppose $(F^4, f_{ij}(x, y))$ is a four dimensional differentiable manifold and $f_{ij}(x, y)$ the anisotropic Finslerian metric is assumed to have signature $(+, -, -, -)$ for any (x, y) .

The motion of a particle in a Finslerian space-time F^4 is described by a pair (x^i, u^i) where $x^i \in F^4$ and $u^i = \frac{dx^i}{d\tau}$, $i = 1, 2, 3, 4$ the 4-velocity of the particle (time-like/null) (τ is proper time) which represents the tangent of its world-line expressing the motion of typical observers in the Finslerian locally anisotropic universe.

A *smooth congruence* in an open coordinate neighborhood U of F^4 can be represented by a preferred family of world lines (time-like curves or null) such that through each couple $(x, u) \in U$ there passes precisely one curve in this family in which u is the tangent vector of this curve to that point x . This consideration is analogous to the Riemannian context. In the framework of a tangent bundle the extended congruence has the form $(x^{(i)}(t), y^{(i)}(t))$. The dependence on a line-element (x, y) can also be considered in this approach [36].

The metric of pseudo-Finslerian space-time is described by the relation

$$ds^2 = F^2(x, y) = f_{ij}y^i y^j$$

The time-like, null and space-like curves can be defined in the Finslerian framework by the following relations [55]

$$\begin{array}{ll} \text{time-like} & f_{ij}(x, y)u^i u^j > 0 \\ \text{null-like} & f_{ij}(x, y)u^i u^j = 0 \\ \text{space-like} & f_{ij}(x, y)u^i u^j < 0 \end{array} \quad (38)$$

In the context of studying of Finsler cosmology has been also considered the form of Riemannian osculation of a metric [25]. The Finslerian metric tensor and a contravariant vector field $y^i(x)$ may be used to construct the Riemannian metric tensor $a_{ij}(x) = g_{ij}(x, y(x))$. The Riemannian space associated with this metric tensor is called the osculating Riemannian space. This gives us the possibility to view some cosmological considerations in a 4-dim space-time framework. In the following we present the content of the Raychaudhuri Equations for a Finsler space-time. We assume Finslerian fluid congruences that the matter flow lines of the fluid are time-like geodesics and are parameterized by the proper time τ so that a vector field $u^i(x)$ of tangents is normalized to the unit length $u^i = \frac{y^i}{F}$.

Using the δ -differentiation in the direction of $u^i(x)$ for a congruence of fluid lines (not necessarily geodesics) are defined the expansion, vorticity and the shear by the following forms: [21]

$$\tilde{\Theta} = Z_{ij}h^{ij} = u_{|i}^i - C_{im}^i \dot{u}^m \quad (39)$$

$$\tilde{\omega}_{ik} = Z_{[ik]} + \dot{u}_i u_k - \dot{u}_k u_i \quad (40)$$

$$\tilde{\sigma}_{ik} = Z_{(ik)} - \frac{1}{3}\tilde{\Theta}h_{ik} - 2C_{ikm}u^m - \dot{u}_i u_k - \dot{u}_k u_i \quad (41)$$

where $\dot{u}^i = u^i_{|k} u^k = Z_k^i u^k$ and “|” denotes the Riemannian covariant derivative associated with the osculating Riemannian metric $a_{ij}(x) = g_{ij}(x, u(x))$. The symbols “[]”, “()” denote the antisymmetrization and symmetrization of Z_{ik} respectively. The tensor $C_{ijk} = \frac{1}{2} \frac{\partial f_{ij}(x, y)}{\partial y^k}$ is symmetric in all its subscripts.

Therefore the extended Finslerian covariant derivative of u can be expressed by

$$Z_{ik} = \frac{1}{3} \tilde{\Theta} h_{ik} + \tilde{\sigma}_{ik} + \tilde{\omega}_{ik} + \dot{u}_i u_k \quad (42)$$

The consideration of a Finslerian incoherent fluid implies that the fluid lines are geodesics and $\dot{u}^i = Z_k^i u^k = 0$. In this case the Finslerian geodesics coincide with the Riemannian ones of a u -Riemannian space (osculating Riemannian).

The commutation relations of δ -covariant derivative of the vector field $u^i(x)$ gives

$$u_{i;hk} - u_{i;kh} = L_{ikh}^j u_j \quad (43)$$

where L_{jkh}^i curvature tensor is derived by the δ -covariant derivative with respect to the osculating affine connection coefficients $a_{jk}^i(x, u(x))$ (Rund, 1959).

$$L_{jkh}^i(x, u(x)) = \left(\frac{\partial L_{jh}^i}{\partial x^k} + \frac{\partial L_{jh}^i}{\partial u^l} \frac{\partial u^l}{\partial x^k} \right) - \left(\frac{\partial L_{jk}^i}{\partial x^h} + \frac{\partial L_{jk}^i}{\partial u^l} \frac{\partial u^l}{\partial x^h} \right) + L_{mk}^i L_{jh}^m - L_{mh}^i L_{jk}^m$$

We finally obtain

$$\frac{d\tilde{\Theta}}{d\tau} = -\frac{1}{3} \tilde{\Theta}^2 - \tilde{\sigma}_{ik} \tilde{\sigma}^{ik} + \tilde{\omega}_{ik} \tilde{\omega}^{ik} - L_{i\ell} u^i u^\ell + \dot{u}_{;i}^i \quad (44)$$

This is *Raychauduri's* equation of the Finslerian space-time. The change of expansion which is expressed by $\frac{d\tilde{\Theta}}{d\tau}$ depends on the V -anisotropic behavior of tensor C_{jk}^i along the matter flow lines. When we consider an incoherent fluid, the fluid-lines are geodesics and the last term of right hand side of (44) is $\dot{u}^i = 0$. In this case the Raychauduri equation is reduced to the form of a u -Riemannian metric of osculating space associated with the congruence of geodesics.

A perfect fluid in the Finslerian space time case has the osculating form

$$T_{ij}(x, u(x)) = (\mu + p) u_i(x) u_j(x) + p a_{ij} \quad (45)$$

where $p = p(x)$, $\mu = \mu(x)$ and $u_i(x)$ represent the pressure, the density of the fluid and the fluid-four velocity respectively.

The Einstein equations can be written in the form

$$L_{ij}(x, u(x)) = K \left(T_{ij}(x, u(x)) - \frac{1}{2} T_k^k a_{ij} \right), \quad K : \text{constant} \quad (46)$$

where the Ricci tensor L_{ij} is directly determined by the matter energy-momentum tensor T_{ij} and at each point associated with the osculating Riemannian metric tensor $a_{ij}(x) = g_{ij}(x, u(x))$. Substitution of (45) to (46) gives

$$L_{ij} u^i u^j = \frac{1}{2} K (\mu + 3p) \quad (47)$$

The term $L_{i\ell} u^i u^\ell$ corresponds to an anisotropic gravitational influence of the matter along the world lines of the fluid and it expresses the tidal force of the field.

From rel.(45) the Raychauduri equation in the case of a perfect fluid is given by

$$\dot{\tilde{\Theta}} = \frac{d\tilde{\Theta}}{d\tau} = -\frac{1}{3} \tilde{\Theta}^2 - \tilde{\sigma}_{ik} \tilde{\sigma}^{ik} + \tilde{\omega}_{ik} \tilde{\omega}^{ik} - \frac{1}{2} K (\mu + 3p) + \dot{u}_{;i}^i \quad (48)$$

The relation

$$L_{i\ell}u^i u^\ell > 0 \quad (49)$$

implies the so called strong energy condition for every time-like vector u^α tangent to time-like geodesics. We notice that the fluid energy μ and pressure p satisfy the energy condition $\mu + p > 0$. This condition uniquely defines the Finslerian world lines (congruences) of the fluid with $u(x)$ tangent vector field analogous to that of Riemannian framework [41] in a region of Finsler space-time which is called "osculating Riemannian manifold" [28]. The term $L_{i\ell}u^i u^\ell > 0$ (rel.46) can also be considered as a key for the existence of conjugate points in the Finslerian space-time structure.

3.1.1 Raychaudhuri Equation in a Finsler-Randers (FR) space

The geodesics

The Lagrangian metric function in a FR space is given by

$$\mathfrak{L} = (g_{ij}\dot{x}^i \dot{x}^j)^{1/2} + k A_i \dot{x}^i, k = \text{constant} \quad (50)$$

where $A_i(x)$ is the electromagnetic potential and $g_{ij}(x)$ the Riemannian gravitational potential with $\dot{x}^i = \frac{dx^i}{ds}$. From (19) we get

$$\begin{aligned} \frac{\partial \mathfrak{L}}{\partial x^a} &= \frac{1}{2(g_{ij}\dot{x}^i \dot{x}^j)^{1/2}} \frac{\partial g_{ij}}{\partial x^a} \dot{x}^i \dot{x}^j + k \frac{\partial A_i}{\partial x^a} \dot{x}^i \\ \frac{\partial \mathfrak{L}}{\partial \dot{x}^a} &= \frac{1}{(g_{ij}\dot{x}^i \dot{x}^j)^{1/2}} g_{aj} \dot{x}^j + k A_a \end{aligned} \quad (51)$$

Thus the Euler–Lagrange equations are derived by the relation

$$\frac{d}{ds} \left(\frac{\partial \mathfrak{L}}{\partial \dot{x}^a} \right) - \frac{\partial \mathfrak{L}}{\partial x^a} = 0 \quad (52)$$

The last equation gives us

$$\frac{d\dot{x}^i}{ds} + \Gamma_{mn}^i \dot{x}^m \dot{x}^n + k(g_{ij}\dot{x}^i \dot{x}^j)^{1/2} F_j^i \dot{x}^j = 0 \quad (53)$$

F_{ik} represents the electromagnetic field. Eq.(53) coincides with the equation of motion of a charged particle in a gravitational field

$$\frac{d\dot{x}^i}{ds} + \Gamma_{mn}^i \dot{x}^m \dot{x}^n = -k F_j^i \dot{x}^j \quad (54)$$

or for a vector V parallel to \dot{x}^m we get

$$\frac{dV^i}{d\tau} + \Gamma_{mn}^i V^m V^n = -k F_j^i V^j, \quad (55)$$

with

$$\dot{V}^i = V_{;k}^i V^k \quad (56)$$

In a FR space the quantity $\dot{V}_{;i}^i \neq 0$ applies in geodesics because the equation of geodesics is the Lorentz force (rel.53). It is incorporated into the Raychaudhuri Equation generalizing the case of the Riemannian ansatz in which the Lorentz Force is introduced in the Raychaudhuri Equation only for non-geodesics flows in the presence of an electromagnetic field [6],[11]. Moreover, in a general Finsler spacetime of an osculating Riemannian form the flow lines follow geodesics motion when the acceleration is equal to zero.

In general we can consider a covector $V_\alpha(x)$ tangent to the flow lines in FR space, then the equation of geodesics is given by

$$\frac{d^2 V^l}{ds^2} + \Gamma_{ij}^l y_i y_j + g^{lm} (\partial_j V_m - \partial_m V_j) y^j = 0 \quad (57)$$

where Γ_{ij}^l represent the Christoffel symbols and g^{lm} the components of the inverse metric of the Riemmanian space-time, $V_m = \phi(x) \widehat{V}_m$ with \widehat{V}_m the unit vector in the direction of V_m and $\phi(x)$ a scalar. We observe that in the equation of geodesics we have an additional term, $g^{lm} (\partial_j V_m - \partial_m V_j) y^j$, which contains a rotation. For the case of electromagnetic waves we must modify the above equation. This is because the world line of an EM wave is null. In geometric optics the direction of propagation of a light ray is determined by the wave vector tangent to the ray. Thus we have:

$$\frac{d V^l}{d\lambda} + \Gamma_{ij}^l \frac{w}{V_i} \frac{w}{V_j} + g^{lm} (\partial_j \frac{w}{V_m} - \partial_m \frac{w}{V_j}) \frac{w}{V^j} = 0 \quad (58)$$

where λ is an affine parameter.

If we substitute the relation $V_{i;j} - V_{j;i} = V_{i,j} - V_{j,i}$ from (57) with the vorticity $\tilde{\omega}_{ij}$ we obtain

$$\frac{dy^\ell}{ds} + \Gamma_{ij}^l y_i y_j + g^{li} \tilde{\omega}_{ij} y^j = 0, y^i = \frac{dx^i}{dt} \quad (59)$$

We note from (57) that the y -dependence of the metric is a consequence of the existence of the anisotropic field V_k in a FR space and the term of vorticity can be included in the equation of geodesics. (rel.59).

The Raychaudhuri equation

Taking into account the presence of an electromagnetic field F_{ik} from rel.(55) the kinematical parameters $\tilde{\Theta}, \tilde{\omega}, \tilde{\sigma}$ of (39)-(41) are modified in the form

$$\tilde{\Theta} = V_{|i}^i + f^m C_{im}^i \quad (60)$$

$$\tilde{\sigma}_{ik} = Z_{(ik)} - \frac{1}{3} \tilde{\Theta} h_{ik} - 2C_{ikm} V^m + \Gamma_{ki0} - \Gamma_{i0k} + 2kF_{ki} \quad (61)$$

$$\tilde{\omega}_{ik} = Z_{[ik]} + \Gamma_{ki0} - \Gamma_{i0k} + 2kF_{ki}. \quad (62)$$

we put $f^m = \Gamma_{jk}^m V^j V^k + kF_j^m V^j$ and $\Gamma_{i0k} = \Gamma_{ijk} V^j$ Here " $|$ " denotes the osculating covariant derivative. In the above equations we used the rel. (55) which is the Lorentz force law in the FR space

By using Cartan's covariant differentiation we get the Cartan curvature tensor \tilde{R}_{jkl}^i and the Ricci tensor \tilde{R}_{jk} . In the framework of a tangent bundle the Raychaudhuri equation has been given in [20]. We can derive the Raychaudhuri equation for the model of FR spacetime (50) in the Finslerian tangent bundle where instead of an electromagnetic potential we can consider a time-like vector field X^n .

$$\begin{aligned} X^m \tilde{\Theta}|_m &= \frac{d\tilde{\Theta}}{d\tau} = \tilde{R}_{lm} X^l X^m - T_{mk}^i X_{|i}^k X^m - \tilde{R}_{mk}^b X^k|_b X^m - X^m|_l X^l|_m = \\ &= \tilde{R}_{km} X^k X^m - T_{mk}^i \left(\frac{1}{3} \tilde{\Theta} \tilde{h}_i^k + \tilde{\sigma}_i^k + \tilde{\omega}_i^k \right) X^m - \tilde{R}_{mk}^b \left(\frac{1}{3} \tilde{\Theta} \tilde{h}_b^k + \sigma_b^k + \omega_b^k \right) X^m - \frac{1}{3} \tilde{\Theta}^2 - \sigma_l^m \sigma_m^l - \omega_l^m \omega_m^l \end{aligned} \quad (63)$$

By considering the rel.(26) and the integrability condition $\tilde{R}_{mk}^b = 0$ of rel.(27) we get from (63) the equation

$$X^m \tilde{\Theta}|_m = \tilde{R}_{km} X^k X^m - \sigma_l^m \sigma_m^l - \omega_l^m \omega_m^l - \frac{1}{3} \tilde{\Theta}^2 \quad (64)$$

Taking into account the form of Ricci tensor \tilde{R}_{kj} in FR space [40],[42] we get

$$\tilde{R}_{kj} = R_{kj} + \Delta_{00k|j} - \Delta_{00j|k} + \Delta_{km0} \Delta_{0j}^m - \Delta_{kmj} \Delta_{00}^m \quad (65)$$

where Δ_{mj}^i represents the deformation tensor (see Appendix A)

$$\Delta_{hmj} = f_{ih} \Delta_{mj}^i \quad (66)$$

R_{kj} is the Ricci tensor of the Riemannian space and f_{ih} represents the metric of FR spacetime. After the rel.(65) the Raychaudhuri equation for the horizontal subspace of the tangent bundle is written

$$X^j \tilde{\Theta}|_j = -\frac{1}{3} \tilde{\Theta}^2 + \{R_{kj} + \Delta_{00k|j} - \Delta_{00j|k} + \Delta_{km0} \Delta_{0j}^m - \Delta_{kmj} \Delta_{00}^m\} X^k X^j - \sigma_l^j \sigma_j^l - \omega_l^j \omega_j^l \quad (67)$$

The form of Raychaudhuri's equation in the vertical space of a FR space can be given by

$$\begin{aligned} X^g \Theta|_g &= \frac{d\Theta}{d\tau} = -\frac{1}{3} \tilde{\Theta}^2 + S_{hg} X^h X^g - S_{ge}^h X^e|_h X^g - X^g|_f X^f|_g = \\ &= -\frac{1}{3} \Theta^2 - \sigma_f^e \sigma_e^f - \omega_f^e \omega_e^f + S_{hg} X^h X^g - S_{ge}^f X^g \left(\frac{1}{3} \Theta h_f^e + \sigma_f^e + \omega_f^e \right) \end{aligned} \quad (68)$$

By imposing $S_{ge}^f = 0$ from the rel(26), the S-Ricci curvature in the FR space gives [47]

$$S_{hg} = \frac{\phi^2}{2F^2\sigma^2} \left[\frac{3\sigma^2(y_h y_g - g_{hg}) - \beta^2(4y_h y_g + 3g_{hg})}{2} + \beta S_{hg}(y_h \widehat{V}_g) \right]$$

where S_{hg} is an operator and denotes symmetrization of the indices h,g we also considered that $\widehat{V}_a \widehat{V}^a = 1$ for timelike vectors and $\sigma = (g_{ij} y^i y^j)^{1/2}$, $\beta = y^a \widehat{V}_a$. Therefore the Raychaudhuri equation in a FR-space takes the form

$$\frac{d\Theta}{d\tau} = -\frac{1}{3} \Theta^2 - \sigma_f^e \sigma_e^f - \omega_f^e \omega_e^f + \left\{ \frac{\phi^2}{2F^2\sigma^2} \left[\frac{3\sigma^2(y_h y_g - g_{hg}) - \beta^2(4y_h y_g + 3g_{hg})}{2} + \beta S_{hg}(y_h \widehat{V}_g) \right] \right\} X^h X^g \quad (69)$$

The form of Raychaudhuri equation in the vertical space can be interpreted as an internal anisotropic contribution in the evolution of the universe.

3.2 Energy Conditions in a FR Space

Energy conditions are of great significance in cosmology because in relation with the Friedmann and the Raychaudhuri equations constitute a key role of universe's evolution [26]. In the following we give the energy conditions in a FR space. In this space an important role is played by the scalar of anisotropy Z_t . The Friedmann-like equations of the generalized form of the FR-type cosmology have been studied in [25]. The form of these equations are given by the relations

$$\frac{\ddot{a}}{a} + \frac{3}{4} \frac{\dot{a}}{a} \dot{u}_0 = -\frac{4\pi G}{3} (\mu + 3P) \quad (70)$$

$$\frac{\ddot{a}}{a} + 2 \frac{\dot{a}^2}{a^2} + 2 \frac{k}{a^2} + \frac{11}{4} \frac{\dot{a}}{a} \dot{u}_0 = 4\pi G (\mu - P) \quad (71)$$

From rel.(70) and (71) we obtain the Friedmann-like equation

$$\left(\frac{\dot{a}}{a} \right)^2 + \frac{\dot{a}}{a} Z_t = \frac{8\pi G}{3} \mu - \frac{k}{a^2} \quad (72)$$

where we set $u_0(t) = \phi(x) \hat{u}_0$, with the time component of the unit vector \hat{u}_a and Z_t defined as $Z_t = \dot{u}_0$. Taking into account relations (70)-(72) we obtain

$$\mu = \frac{3}{8\pi G} \left(\frac{\dot{a}^2}{a^2} + \frac{\dot{a}}{a} Z_t + \frac{k}{a^2} \right) = \frac{3}{8\pi G} (H^2 + \frac{k}{a^2} + Z_t H) \quad (73)$$

with $H = \frac{\dot{a}}{a}$. From (73) in order to be valid $\mu \geq 0$ the scalar should be $Z_t \geq 0$. We also have

$$P = -\frac{1}{8\pi G} \left[2 \left(\frac{\ddot{a}}{a} \right) + H^2 + \left(\frac{k}{a^2} \right) + \frac{5Z_t}{2} H \right] \quad (74)$$

From(70)-(74) we obtain

$$\mu + P = \frac{1}{8\pi G} \left[-2\frac{\ddot{a}}{a} + 2H^2 + 2\frac{k}{a^2} + \frac{1}{2}Z_t H \right] \quad (75)$$

and

$$\mu + 3P = \frac{-3}{4\pi G} \left[\frac{\ddot{a}}{a} + \frac{3}{4}\frac{\dot{a}}{a}Z_t \right] \quad (76)$$

The strong energy condition $\mu + 3P \geq 0$ is obtained for $\frac{\ddot{a}}{a} \leq -\frac{3}{4}Z_t$ consequently $\ddot{a} < 0 (Z_t > 0)$. We also have $\mu + P \geq 0$ if $\ddot{a} < 0$ from (75). In addition the weak energy conditions (WEC) and the null energy conditions (SEC) are also satisfied.

A bounce occurs in the universe when WEC, NEC, SEC are violated for a short interval of a point time (bounce time) with $\dot{a} = 0$ and $\ddot{a} > 0$ [43], [44], [45]. In analogy with FRW-universe for a FR-cosmology a bounce can be considered if the above mentioned conditions for the WEC and SEC are violated. In this case $HZ_t = 0$ and $\ddot{a} > 0$. In a bounce time the SEC conditions of a FR spacetime and of a FRW-cosmology one are identified. We notice that the bounce conditions $\mu + P < 0, \mu + 3P < 0$ are equivalent to

$$2(H^2 + \frac{k}{a^2}) + \frac{1}{2}HZ_t < 2\frac{\ddot{a}}{a} \Rightarrow \ddot{a} > 0 \quad (77)$$

$$\ddot{a} > -\frac{3}{4}Z_t\dot{a} \Rightarrow \ddot{a} > 0 \quad (78)$$

4 Raychaudhuri Equation in Generalized Scalar-Tensor Theory

4.1 The model

Scalar-tensor theories constitute a fundamental part of studying of general relativity and cosmology. The significance of scalar-tensor and Brans-Dicke theories to the cosmology has mainly pointed out in [49],[53] et all. On the other hand, the simplest standard models of inflation involve one scalar field. However in string or supergravity theories it is possible one to study models with several different scalar fields especially if they have some new properties. The mechanism that two scalars $\sigma^{(1)}, \sigma^{(2)}$ drive the inflation in a more complicated theory is useful for studying of the evolution of the universe[54]. When the standard mechanism does not work, two scalars can give us an additional freedom in finding realistic models of inflationary cosmology.

In the framework of generalized metric structures of scalar-tensor theory of gravitation a Langrangian density has been studied [39]. Furthermore, the field equations for the Finslerian gravitational field in a space of the form $M \times \{\phi^{(1)}\} \times \{\phi^{(2)}\}$ are derived where M represents a pseudo-Riemannian space-time manifold with $\phi^{(1)}, \phi^{(2)}$ two non-commutative G-numbers(Grassmannian). In our case we consider a 4-pseudo-Riemannian manifold coupled with two scalars $\phi^{(1)}, \phi^{(2)}$ playing the role of fibres or the internal variables. Physically these can represent the inflaton and an anisotropy come from a scalar field.[24]

In this space the adapted frame has the form $\frac{\partial}{\partial Z^M} \equiv \left(\frac{\delta}{\delta x^\alpha} = \frac{\partial}{\partial x^\alpha} - N_\alpha^{(1)} \frac{\partial}{\partial \phi^{(1)}} - N_\alpha^{(2)} \frac{\partial}{\partial \phi^{(2)}}, \frac{\partial}{\partial \phi^{(1)}}, \frac{\partial}{\partial \phi^{(2)}} \right)$

$$dZ^M \equiv (dx^\alpha, \delta\phi^{(1)} = d\phi^{(1)} + N_\alpha^{(1)} dx^\alpha, \delta\phi^{(2)} = d\phi^{(2)} + N_\alpha^{(2)} dx^\alpha)$$

We put $X_\alpha = \frac{\delta}{\delta x^\alpha}, X_{(1)} = \frac{\partial}{\partial \phi^{(1)}}, X_{(2)} = \frac{\partial}{\partial \phi^{(2)}}$ and $X_M = X_\alpha, X_{(1)}, X_{(2)}$. As well we consider the condition

$$\phi(x^a) = g_{(1)(1)}(x^a) = g_{(2)(2)}(x^a), \text{ with } \phi : \text{scalar and } a = 0, 1, 2, 3.$$

The geometrical concepts $N_a^{(1)}, N_a^{(2)}$ represent the non-linear connections of scalar fields $\phi^{(1)}, \phi^{(2)}$ with respect to the space-time coordinates.

The inverse metric of G_{MN} is given by $G^{MN} = [g^{\alpha\beta}, g^{(1)(1)}, g^{(2)(2)}]$ where $g^{\alpha\beta}$ is the inverse metric of $g_{\alpha\beta}$ for M. The covector X_M has the inverse $X^N = G^{MN} X_M$.

$$X^{(1)} = \phi^{-1} X_{(1)} = g^{(1)(1)} X_1$$

$$X^{(2)} = \phi^{-1} X_{(2)} = g^{(2)(2)} X_{(2)}$$

$$\phi^{-1} = g^{(1)(1)} = g^{(2)(2)}$$

The metric structure of this model concerning the adapted basis is given by

$$G = G_{MN} dZ^M dZ^N = g_{\alpha\beta} dx^\alpha \otimes dx^\beta + g_{(1)(1)} \delta\phi^{(1)} \otimes \delta\phi^{(1)} + g_{(2)(2)} \delta\phi^{(2)} \otimes \delta\phi^{(2)} \quad (79)$$

The curvature tensor in this space is written

$$R_{LNM}^K = X_M \Gamma_{LN}^K - X_N \Gamma_{LM}^K + \Gamma_{LN}^Z \Gamma_{ZM}^K - \Gamma_{LM}^Z \Gamma_{ZN}^K + \Gamma_{LZ}^K W_{NM}^Z \quad (80)$$

In our approach, we shall adopt the case where all the coefficients of torsion tensors are equal to zero and $g_{a(1)} = g_{(1)a} = g_{a(2)} = g_{(2)a} = g_{(1)(2)} = g_{(2)(1)} = 0$. The Ricci tensor in this model is given by

$$R_{MN} \equiv R_{MLN}^L \equiv (R_{\alpha\beta}, \dots, R_{(1)(1)}, R_{(2)(2)}) \quad (81)$$

or

$$R_{MN} = \{R_{MN\alpha}^\alpha, R_{MN(1)}^{(1)}, R_{MN(2)}^{(2)}\} \quad (82)$$

$$\mathcal{R} = R_{MN} G^{MN} = R_{\alpha\beta} g^{\alpha\beta} + R_{(1)(1)} g^{(1)(1)} + R_{(2)(2)} g^{(2)(2)} = R + R^{(1)} + R^{(2)} \quad (83)$$

$G_{MN} = (g_{\beta\gamma}(x^\alpha), (g_{(1)(1)}(x^\alpha), (g_{(2)(2)}(x^\alpha))$ with $(g_{(1)(1)}(x^\alpha) = (g_{(2)(2)}(x^\alpha) = \phi(x^\alpha) \neq 0$ is a scalar function. In our case the Ricci tensors take the form

$$\tilde{R}_{\alpha\beta} = R_{\alpha\beta} + \Gamma_{\alpha\beta}^\mu \phi_{,\mu} \phi^{-1} + \frac{1}{2} \phi_{,\alpha} \phi_{,\beta} \phi^{-1} - \phi_{,\alpha\beta} \phi^{-1} \quad (84)$$

$$\tilde{R}_{(1)(1)} = R_{(2)(2)} = \frac{1}{2} g^{\alpha\mu} g^{\beta\nu} g_{\alpha\beta,\mu} \phi_{,\nu} - \frac{1}{2} g^{\mu\nu} (\phi_{,\mu\nu} + \frac{1}{2} g_{,\mu} \phi_{,\nu}) \quad (85)$$

The indices (α, β, \dots) run over $(0, \dots, 3)$ and $(1), (2)$ represent the fibres. In this model the Langrangian density is given by [39]

$$L = \sqrt{|g|} (\phi R - 2g^{\alpha\beta} \phi_{|\alpha\beta} + \frac{1}{4} g^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} / \phi)$$

Varying the action (L) with respect to $g_{\mu\nu}$ and ϕ we obtain the equations

$$\frac{\delta L}{\delta g_{\kappa\lambda}} \equiv \sqrt{|g|} E^{\kappa\lambda} + \frac{1}{4} \sqrt{|g|} (g^{\alpha\beta} g^{\kappa\lambda} \phi^{-1} - g^{\alpha\kappa} g^{\beta\lambda}) \phi_{,\alpha} \phi_{,\beta} + \sqrt{|g|} \phi_{|\alpha\beta} (g^{\alpha\kappa} g^{\beta\lambda} - g^{\alpha\beta} g^{\kappa\lambda}) = 0 \quad (86)$$

$$L_\phi = \frac{\delta L}{\delta \phi} \equiv \sqrt{|g|} \left(\phi R + \frac{1}{2} g^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} \phi^{-1} - g^{\alpha\beta} \phi_{|\alpha\beta} \right) = 0 \quad (87)$$

where $E^{\kappa\lambda} = R^{\kappa\lambda} - \frac{1}{2} g^{\kappa\lambda}$. The stress-energy tensor can be defined by the form $T^{\kappa\lambda} = -2 \frac{1}{\sqrt{|g|}} \frac{\delta L}{\delta g_{\kappa\lambda}}$. In addition rel.(87) provides us with the energy-momentum tensor for the inflaton field in the form $T^{\mu\nu}(\phi) = -2 \frac{1}{\sqrt{|g|}} \frac{\delta L_\phi}{\delta g_{\mu\nu}}$

4.2 The Raychaudhuri equation of the model

If we substitute the relations (84)(85) in this model we obtain the Raychaudhuri equation

$$X^N \tilde{\Theta}|_N = \frac{d\tilde{\Theta}}{d\tau} = \mathcal{R}_{MN} X^M X^N - \frac{1}{3} \tilde{\Theta}^2 - \sigma_K^N \sigma_N^K - \omega_K^N \omega_N^K$$

or $X^N \tilde{\Theta}|_N =$

$$= \{R_{\alpha\beta} + \Gamma_{\alpha\beta}^\mu \phi_{,\mu} \phi^{-1} + \frac{1}{2} \phi_{,\alpha} \phi_{,\beta} \phi^{-1} - \phi_{,\alpha\beta} \phi^{-1} + g^{\alpha\mu} g^{\beta\nu} g_{\alpha\beta,\mu} \phi_{,\nu} - \frac{1}{2} g^{\mu\nu} (\phi_{,\mu\nu} + \frac{1}{2} g_{,\mu} \phi_{,\nu})\} X^M X^N$$

$$-\frac{1}{3}\tilde{\Theta}^2 - \sigma_K^N \sigma_N^K - \omega_K^N \omega_N^K \quad (88)$$

where we have considered the conditions $T_{mk}^i = R_{mk}^b = 0$ and \tilde{R}_{MN} are given by .

The symbols " $|$ ", " $|$ " denote the horizontal and vertical covariant derivatives ref.[37], [38]. We can apply the metric of FRW geometry with coordinates $(t, r, \theta, \phi) = (x^a)$ for the space-time $M \times \phi^{(1)} \times \phi^{(2)}$. We use the Ricci tensors

$$\begin{aligned} R_{00} &= 3\frac{\ddot{a}}{a} = 4\pi G(\mu + 3p) \\ R_{11} &= -\frac{a\ddot{a} + 2\dot{a}^2 + 2k}{1 - kr^2} \\ R_{22} &= -\frac{(a\ddot{a} + 2\dot{a}^2 + 2k)}{r^2} \\ R_{33} &= -\frac{(a\ddot{a} + 2\dot{a}^2 + 2k)}{r^2 \sin^2 \theta}, \end{aligned}$$

$$\begin{aligned} \tilde{R}_{MN} &= R_{00} + R_{11} + R_{22} + R_{33} + R_{(1)(1)} + R_{(2)(2)} = \\ &= 4\pi G(\mu + 3p) + \frac{1}{2}(\phi_{,t})^2 \phi^{-1} - \phi_{,tt} \phi^{-1} + \frac{a\ddot{a} + 2\dot{a}^2 + 2k}{1 - kr^2} + \frac{k}{2}(1 - kr)^{-1} \phi_{,r} \phi^{-1} + \frac{1}{2} \phi_{,r}^2 \phi^{-1} - \phi_{,rr} \phi^{-1} - \\ &\quad - \frac{a\ddot{a} + 2\dot{a}^2 + 2k}{r^2} + \frac{1}{2} \phi_{,\theta}^2 \phi^{-1} - \phi_{,\theta\theta} \phi^{-1} - \frac{a\ddot{a} + 2\dot{a}^2 + 2k}{r^2 \sin^2 \theta} + \frac{1}{2} \phi_{,\phi}^2 \phi^{-1} - \phi_{,\phi\phi} \phi^{-1} + \phi_{,tt} + a^{-2} k \phi_{,r} - \\ &\quad - \frac{1 - kr^2}{a^4} \phi_{,rr} + \frac{(4 - 2kr)r^3 a^6 \sin^2 \theta}{(1 - kr)^2} \phi_{,r} - (ar)^{-2} \phi_{,\theta\theta} + \left(\frac{r^4 a^6 \sin^2 \theta}{1 - kr^2}\right)_{,\theta} \phi_{,\theta} (ar)^{-2} - (ar \sin \theta)^{-2} \phi_{,\phi\phi} \end{aligned} \quad (89)$$

From (88) and (89) the tidal force field with $\tilde{R}_{MN} X^M X^N$ is written

$$\tilde{R}_{MN} X^M X^N = \tilde{\mathcal{A}}_{MN} X^M X^N \quad (90)$$

where we let

$$\begin{aligned} \tilde{\mathcal{A}}_{MN} &= [4\pi G(\mu + 3p) + \frac{1}{2}(\phi_{,t})^2 \phi^{-1} - \phi_{,tt} \phi^{-1} + \frac{a\ddot{a} + 2\dot{a}^2 + 2k}{1 - kr^2} + \frac{k}{2}(1 - kr)^{-1} \phi_{,r} \phi^{-1} + \frac{1}{2} \phi_{,r}^2 \phi^{-1} - \phi_{,rr} \phi^{-1} - \\ &\quad - \frac{a\ddot{a} + 2\dot{a}^2 + 2k}{r^2} + \frac{1}{2} \phi_{,\theta}^2 \phi^{-1} - \phi_{,\theta\theta} \phi^{-1} - \frac{a\ddot{a} + 2\dot{a}^2 + 2k}{r^2 \sin^2 \theta} + \frac{1}{2} \phi_{,\phi}^2 \phi^{-1} - \phi_{,\phi\phi} \phi^{-1} + \phi_{,tt} + a^{-2} k \phi_{,r} - \\ &\quad - \frac{1 - kr^2}{a^4} \phi_{,rr} + \frac{(4 - 2kr)r^3 a^6 \sin^2 \theta}{(1 - kr)^2} \phi_{,r} - (ar)^{-2} \phi_{,\theta\theta} + \left(\frac{r^4 a^6 \sin^2 \theta}{1 - kr^2}\right)_{,\theta} \phi_{,\theta} (ar)^{-2} - (ar \sin \theta)^{-2} \phi_{,\phi\phi}] \end{aligned} \quad (91)$$

Therefore the Raychaudhuri equation takes the form

$$X^N \tilde{\Theta}|_N = \frac{d\tilde{\Theta}}{d\tau} = \tilde{\mathcal{A}}_{MN} X^M X^N - \frac{1}{3}\tilde{\Theta}^2 - \sigma_K^N \sigma_N^K - \omega_K^N \omega_N^K \quad (92)$$

If $\tilde{\mathcal{A}}_{MN} > 0$, it contributes to defocusing or acceleration, whereas if $\tilde{\mathcal{A}}_{MN} < 0$, we have focusing or deceleration to the evolution of the universe.

4.3 Contribution of non-linear connection to the equation

In a more general case, if we consider the torsion (rel(6)) we obtain the Ricci tensors $K_{\alpha\beta}, K_{(1)(1)}, K_{(2)(2)}$:

$$K_{\alpha\beta} = \tilde{R}_{\alpha\beta} + \frac{1}{2}\phi_{,\alpha}\phi^{-1} \left(\frac{\partial N_{\beta}^{(1)}}{\partial\phi^{(1)}} + \frac{\partial N_{\beta}^{(2)}}{\partial\phi^{(2)}} \right) \quad (93)$$

$$K_{(1)(1)} = R_{(1)(1)} + \frac{1}{2}g^{\mu\nu}\phi_{,\nu} \left(\frac{\partial N_{\mu}^{(1)}}{\partial\phi^{(1)}} \right) \quad (94)$$

$$K_{(2)(2)} = R_{(2)(2)} + \frac{1}{2}g^{\mu\nu}\phi_{,\nu} \left(\frac{\partial N_{\mu}^{(2)}}{\partial\phi^{(2)}} \right) \quad (95)$$

where $\tilde{R}_{\alpha\beta}, R_{(1)(1)}, R_{(2)(2)}$ are given by (84),(85).

The non-linear connections $N_{\beta}^{(1)}, N_{\beta}^{(2)}$ can physically represent the interaction field between space-time and inflaton or anisotropy. In addition, the terms $\frac{\partial N_a^{(1)}}{\partial\phi^{(1)}}, \frac{\partial N_a^{(2)}}{\partial\phi^{(2)}}$ can be interpreted as the variation of the interactive fields with respect to local anisotropy of space-time or the inflaton. In this case the field equations produced by the variational principle of the generalized Lagrangian [39]

$$\tilde{\mathcal{L}} = \sqrt{|G|}G^{AB}K_{AB}$$

given by

$$\delta\tilde{\mathcal{L}}_{\phi} = \frac{\delta\tilde{\mathcal{L}}}{\delta\phi} = \sqrt{|g|}g^{\alpha\beta} \left(R_{\alpha\beta} - \frac{\phi_{;\alpha\beta}}{\phi} + \frac{\phi_{;\alpha}\phi_{;\beta}}{2\phi^2} \right) - \frac{\delta}{\delta x^a} \left[\sqrt{|g|}g^{\alpha\beta} \left(\frac{\partial N_{\beta}^{(1)}}{\partial\phi^{(1)}} + \frac{\partial N_{\beta}^{(2)}}{\partial\phi^{(2)}} \right) \right] \quad (96)$$

where $\mathcal{K}_{AB} = \{K_{\alpha\beta}, K_{(1)(1)}, K_{(2)(2)}\}$. The previous equation can define the energy-momentum tensor $\tilde{T}^{\kappa\lambda}(\phi)$ for the inflaton field with the contribution of non-linear connections terms $N_a^{(1)}, N_a^{(2)}$ as

$$\tilde{T}^{\kappa\lambda}(\phi) = -2 \frac{1}{|g|} \frac{\delta\tilde{\mathcal{L}}_{\phi}}{\delta g_{\kappa\lambda}} \quad (97)$$

In this structure of spacetime the Raychaudhuri equation is written as

$$X^N \tilde{\Theta}|_N = \frac{d\tilde{\Theta}}{d\tau} = \mathcal{K}_{LN}X^L X^N - \frac{1}{3}\tilde{\Theta}^2 - \sigma_K^N \sigma_N^K - \omega_K^N \omega_N^K \quad (98)$$

Following through the same procedure as above we can obtain the Raychaudhuri equations with the contribution of non-linear connection.

$$X^N \tilde{\Theta}|_N = \frac{d\tilde{\Theta}}{d\tau} = \tilde{\mathcal{B}}_{MN}X^M X^N - \frac{1}{3}\tilde{\Theta}^2 - \sigma_K^N \sigma_N^K - \omega_K^N \omega_N^K \quad (99)$$

where

$$\tilde{\mathcal{B}}_{MN} = \tilde{\mathcal{A}}_{MN} + \frac{1}{2}\phi_{,\alpha}\phi^{-1} \left(\frac{\partial N_{\beta}^{(1)}}{\partial\phi^{(1)}} + \frac{\partial N_{\beta}^{(2)}}{\partial\phi^{(2)}} \right) + \frac{1}{2}g^{\mu\nu}\phi_{,\nu} \left(\frac{\partial N_{\mu}^{(1)}}{\partial\phi^{(1)}} \right) + \frac{1}{2}g^{\mu\nu}\phi_{,\nu} \left(\frac{\partial N_{\mu}^{(2)}}{\partial\phi^{(2)}} \right) \quad (100)$$

and $\tilde{\mathcal{A}}_{MN}$ was given by (91). From (99),(100) it seems that the interaction terms can contribute to decelerate form of space-time.

5 Conclusions

The fundamental role of Raychaudhuri equation for the general relativity is extended in the framework of generalized geometrical structures of a Finsler-Randers spacetime and of generalized scalar-tensor theories. Additional terms in the equation are introduced because of the anisotropic curvatures and of the form of geodesics (rel.(59)). In this approach we used the Cartan connection since it preserves the norm of a vector (time-like, null) under a parallel propagation and it is convenient for studying of modified gravitational and cosmological theories. The Raychaudhuri equation was also derived in connection with the FRW model for the FR space-time. In par.3 we studied the energy conditions for the FR-cosmology and their relations with the FRW-cosmology. In addition we provide the bounce conditions for a FR cosmology. It was proved that in a bounce both the energy conditions are identified. The study of Raychaudhuri equation in a generalized scalar-tensor theory of a model $M \times \{\phi^{(1)}\} \times \{\phi^{(2)}\}$ with the presence of non-linear connections as extra terms can play a significant role to the gravitational influence and its interaction with other fields. This means that these additional terms/fields will differentiate the evolution of accelerated expansion of the universe (focusing/defocusing).

6 Appendix A

The Lagrangian metric function in a Finsler-Randers (FR) space is given by

$$L(x, y) = (g_{ij}(x)y^i y^j)^{1/2} + a_i(x)y^i$$

where a_i represent a covector and g_{ij} the metric tensor of the pseudo-Riemannian spacetime. By using Cartan's connection the deformation tensor Δ_{jk}^i in a FR space is defined as

$$\Delta_{jk}^i = \Gamma_{jk}^i - L_{jk}^i, \quad (A.1)$$

[40],[42],[47] where Γ_{jk}^i are the Christoffel symbols of Riemannian space and L_{jk}^i the Cartan connections coefficients of FR space. The contraction terms $\Delta_{j0}^i, \Delta_{00}^i$ are given by

$$\Delta_{j0}^i = p^i a_{(j0)} + \frac{1}{2} h_j^i a_{00} + G^{is} (a_{[sj]} + a_{[s0]} p_j) - 2g^{ms} A_{jm}^i a_{[s0]} / \tau, \quad (A.2)$$

$$\Delta_{00}^i = 2(\lambda p^i + g^{is} a_{[s0]} / \tau), \quad (A.3)$$

where the torsion tensor A_{jk}^i is defined by

$$A_{jk}^i = (h_j^i L_k + h_k^i L_j + \frac{2}{5} h_{jk} A^i) / 2,$$

and

$$a_{ij} = \nabla_j a_i,$$

with ∇_j we denote the covariant derivative with respect to the Riemannian space. Also the symbols $(), [],$ mean symmetrization, antisymmetrization and "o" represents contraction with p^i

$$h_{ij} = \tau(g_{ij} - a_i a_j)$$

$$h_j^i = G^{ik} h_{kj} = \delta_j^i - p^i p_j,$$

G^{ij} denote the components of the inverse metric of FR space and

$$\tau = L / g^{1/2}$$

$$p^i = y^i / L$$

$$p_i = \partial L / \partial Y^i$$

$$L_i = a_i - \mu l_i$$

$$\begin{aligned}\mu &= a_i l^i \\ \ell^i &= y^i / g^{1/2}\end{aligned}$$

If $\Delta_{jk}^i = 0$ from rel.(A.1), L_{jk}^i are the Christoffel symbols Γ_{jk}^i of the Riemannian space. In this case the Randers space is called a Landsberg-Randers space. The h-curvature \tilde{R}_{hjk}^i of FR space is connected with the Riemannian one R_{hjk}^i with the relation

$$\tilde{R}_{hjk}^i = R_{hjk}^i + A_{hr}^i R_{0jk}^r, \quad (A.4)$$

In addition if the torsion tensor $A_{hr}^i = 0$ the Finsler Randers spaces reduces in a Riemannian one.

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